Discrete Mathematics 312 (2012) 2190–2196

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

An implicit degree condition for hamiltonian graphs

Hao Li [a,b,*], Wantao Ning [a,c], Junqing Cai [a]

[a] *School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, PR China*
[b] *L R I, UMR 8623 CNRS and Université de Paris-Sud 11, F-91405 Orsay, France*
[c] *Department of Mathematics, Xidian University, Xi'an, Shaanxi 710071, PR China*

A R T I C L E I N F O

Article history:
Received 23 November 2010
Accepted 1 March 2012
Available online 28 March 2012

Keywords:
Implicit degree
Independent set
Hamiltonian cycle
Degree sum

A B S T R A C T

In 1980, Bondy generalized known Ore's theorem by proving that a k-connected graph of order n is hamiltonian if a degree sum of any $k+1$ independent vertices is greater than $(k+1)(n-1)/2$. In this work, we generalize this result replacing the degree sum by the implicit degree sum. A concept of the implicit degree was introduced by Zhu et al. in 1989 [5].

1. Introduction

All graphs considered here are undirected and simple. We use [2] for terminology and notation not defined here. Throughout this paper, G is a graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)| = n$. For $v \in V(G)$, we denote by $N(v)$ the neighborhood of v in G, and $d(v) = |N(v)|$ is the degree of v in G. Let M be a subgraph of G. Then $N_M(v) = N(v) \cap V(M)$. Let

$$\sigma_k(G) = \min\left\{\sum_{i=1}^{k} d(x_i) : x_1, x_2, \ldots, x_k \text{ are } k \text{ independent vertices of } G\right\}.$$

If $S \subseteq V(G)$, $G - S$ means the graph obtained from G by deleting all the vertices of S and all the edges with an endpoint in S. For two vertex disjoint graphs H and Q, the *union* $H \cup Q$ of H and Q is the graph with vertex set $V(H) \cup V(Q)$ and edge set $E(H) \cup E(Q)$. We use mQ instead of $\bigcup_{i=1}^{m} Q_i$, where each Q_i is isomorphic to Q. The *join* $H + Q$ of H and Q is the graph obtained from $H \cup Q$ by joining each vertex of H to each vertex of Q.

A graph G is *Hamiltonian* if G has a cycle containing all vertices of G. This cycle is called a *Hamiltonian cycle*. The following theorem due to Dirac is a basic result of the extremal Hamiltonian graph theory.

Theorem 1.1 (*Dirac [3]*). *Let G be a graph on $n \geq 3$ vertices. If $\sigma_1(G) \geq n/2$, then G is Hamiltonian.*

This result was generalized by Ore who studied a degree sum condition instead of a minimum degree condition in Dirac's theorem.

Theorem 1.2 (*Ore [4]*). *Let G be a graph on $n \geq 3$ vertices. If $\sigma_2(G) \geq n$, then G is Hamiltonian.*

* Corresponding author at: School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, PR China.
E-mail addresses: li@lri.fr (H. Li), ningwt05@163.com (W. Ning), cajq09@lzu.edu.cn (J. Cai).

0012-365X/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2012.03.001

More general, Bondy extended these results by considering a degree sum condition of more independent vertices.

Theorem 1.3 (*Bondy [1]*). *Let G be a k-connected graph of order $n \geq 3$. If $\sigma_{k+1}(G) > (k+1)(n-1)/2$, then G is Hamiltonian.*

The above results refer to classical degrees of vertices of a graph. In this paper, we are interested in, so called, implicit degrees. Define $N_i(v) = \{u \in V(G) : d(u, v) = i\}$, where $i = 1, 2$ and $d(u, v)$ is the distance between u and v.

Definition 1.1 (*Zhu et al. [5]*). Let $v \in V(G)$.

(a) If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, denote by $M_2 = \max\{d(u) : u \in N_2(v)\}$ and by $m_2 = \min\{d(u) : u \in N_2(v)\}$. Let $d(v) = l+1$ and $d_1 \leq d_2 \leq d_3 \leq \cdots \leq d_l \leq d_{l+1} \leq \cdots$ be the degree sequence of the vertices of $N_1(v) \cup N_2(v)$. Put

$$d^*(v) = \begin{cases} m_2, & \text{if } d_l < m_2; \\ d_{l+1}, & \text{if } d_{l+1} > M_2; \\ d_l, & \text{if } d_l \geq m_2 \text{ and } d_{l+1} \leq M_2. \end{cases}$$

Then the implicit degree $id(v)$ of v is defined as $\max\{d(v), d^*(v)\}$.

(b) If $N_2(v) = \emptyset$ or $d(v) < 2$, then $id(v) = d(v)$.

It is clear from the definition that $id(v) \geq d(v)$ for every vertex v.

Let

$$\sigma_k^*(G) = \min\left\{\sum_{i=1}^{k} id(x_i) : x_1, x_2, \ldots, x_k \text{ are } k \text{ independent vertices of } G\right\}.$$

In this paper, we show the following new result which is a Bondy type condition (see Theorem 1.3) with implicit degrees.

Theorem 1.4. *Let G be a k-connected graph of order $n \geq 3$. If $\sigma_{k+1}^*(G) > (k+1)(n-1)/2$, then G is Hamiltonian.*

The proof of Theorem 1.4 will be given in the next section. Now we present the following examples. One shows the sharpness of Theorem 1.4 and the other shows a graph which does not satisfy the conditions of Theorem 1.3 but it can be easily verified to be Hamiltonian by using Theorem 1.4.

Example 1.1. Let k be a nonnegative integer. Let $G = G[X, Y]$ be a complete bipartite graph with color classes such that $|X| = m$, $|Y| = m+1$ and $m \geq k$. Then $|V(G)| = n = 2m+1$. Obviously G is not Hamiltonian. For any $u \in X$ and $v \in Y$, we have $id(u) = m+1$ and $id(v) = m$. Hence $\sigma_{k+1}^*(G) = (k+1)m = (k+1)((m+m+1)-1)/2 = (k+1)(n-1)/2$, which implies that the lower bound of Theorem 1.4 is best possible.

Example 1.2. Let k be a nonnegative integer. For any $m \geq k+1$, let $G := K_{2m} + (K_4 \cup mK_2)$. We can easily verify that G is Hamiltonian. The order of G is $n = 4m+4$ and G is $2m$-connected. We have $d(u) = 2m+1$ and $id(u) = 2m+3$ for any vertex u in mK_2, $d(v) = id(v) = 4m+3$ for any vertex v in K_{2m}, and $d(w) = id(w) = 2m+3$ for any vertex w in K_4. It follows that for any $k+1$ independent vertices $x_1, x_2, \ldots, x_{k+1}$ in mK_2, $\sum_{i=1}^{k+1} d(x_i) = (k+1)(2m+1) < (k+1)(n-1)/2$. Hence we cannot obtain the hamiltonicity of G from Theorem 1.3. However, for any $k+1$ independent vertices $x_1, x_2, \ldots, x_k, y$ with $x_1, x_2, \ldots, x_k$ in mK_2 and y in K_4 or mK_2, $id(y) + \sum_{i=1}^{k} id(x_i) = (k+1)(2m+3) > (k+1)(n-1)/2$. It gives $\sigma_{k+1}^*(G) > (k+1)(n-1)/2$ and we can apply Theorem 1.4 to the graph G.

2. Proof of the main theorem

Let $C = v_1v_2 \ldots v_kv_1$ be a cycle of G with a fixed orientation. We denote by v_i^+ and v_i^- the successor and predecessor of v_i on C, respectively. Let x and y be two vertices of C, the subpath $xx^+ \cdots y^-y$ of C is denoted by $C(x, y)$ (or $x\vec{C}y$). The same path, in reverse order, is denoted by $\bar{C}(y, x)$ (or $y\bar{C}x$). Let $P_1 = u_1u_2 \ldots u_k$ and $P_2 = v_1v_2 \ldots v_d$ be two paths. If $V(P_1) \cap V(P_2) = \{u_k\} = \{v_1\}$, set $P_1 \cup P_2 = u_1u_2 \ldots u_kv_2v_3 \ldots v_d$. If $V(P_1) \cap V(P_2) = \emptyset$ and $u_kv_1 \in E(G)$, set $P_1 \cup P_2 = u_1u_2 \ldots u_kv_1v_2v_3 \ldots v_d$. We also regard a vertex as a path.

Let $P(a, b) = v_0v_1v_2 \ldots v_l$, with $v_0 = a$ and $v_l = b$, be a path connecting a and b in G. For $0 \leq i \leq j \leq l$, the subpath of P from v_i to v_j is denoted by $P(v_i, v_j)$. Similarly, the same path, in reverse order, is denoted by $\bar{P}(v_j, v_i)$. Let

$$l_P(a) = \max\{i : v_i \in V(P) \text{ and } v_ia \in E(G)\},$$

and

$$l_P(b) = \min\{i : v_i \in V(P) \text{ and } v_ib \in E(G)\}.$$

Denote $L_P(a) = v_{l_P(a)}$ and $L_P(b) = v_{l_P(b)}$. For a vertex $v_i \in V(P)$, we define

$$N_P^-(v_i) = \{v_j : v_{j+1}v_i \in E(G)\} \quad \text{and} \quad N_P^+(v_i) = \{v_j : v_{j-1}v_i \in E(G)\}.$$

Proof of Theorem 1.4. Suppose that G is a graph satisfying the conditions of Theorem 1.4 and G is not Hamiltonian. Let C be a longest cycle of G. Then $|V(C)| < n$ and let H be a component of $G - V(C)$. Take a vertex $u_0 \in V(H)$. Since G is k-connected, there are k paths $P_1(u_0, x_1), P_2(u_0, x_2), \ldots, P_k(u_0, x_k)$ from u_0 to C such that $V(P_i) \cap V(C) = \{x_i\}$ for each i with $1 \le i \le k$ and $V(P_i) \cap V(P_j) = \{u_0\}$ for each i and j with $1 \le i \ne j \le k$. We may assume $x_1, x_2, \ldots, x_k$ occur on C in the order of their indices. Let $u_i = x_i^+$ for $i \in \{1, 2, \ldots, k\}$. Throughout the proof, the indices will be taken modulo $k+1$.

The following claim is a simple consequence of the choice of the cycle C.

Claim 2.1. *For any i and j with $0 \le i < j \le k$, there is no path in $G-(V(C)\setminus\{u_i, u_j\})$ connecting u_i and u_j. Hence $\{u_0, u_1, \ldots, u_k\}$ is an independent set.* $\square$

Put

$$F_0(u_0, u_1) = P_1(u_0, x_1) \cup \bar{C}(x_1, u_1) \quad \text{and} \quad F_k(u_k, u_0) = C(u_k, x_k) \cup \bar{P}_k(x_k, u_0),$$

and for any i with $1 \le i \le k-1$,

$$F_i(u_i, u_{i+1}) = C(u_i, x_{i+1}) \cup \bar{P}_{i+1}(x_{i+1}, u_0) \cup P_i(u_0, x_i) \cup \bar{C}(x_i, u_{i+1}).$$

Note that, for any i with $0 \le i \le k$, $V(C) \cup \{u_0\} \subseteq V(F_i)$ and then $|V(F_i)| \ge |V(C)| + 1$.

For $j \in \{1, 2, \ldots, k\}$, let $F'_j(u'_j, u_{j+1})$ be a path containing $F_j(u_j, u_{j+1})$ as a subpath such that

(i) $F'_j(u'_j, u_{j+1})$ is as long as possible,
(ii) $l_{F'_j}(u'_j)$ is as large as possible, subject to (i).

For $j \in \{1, 2, \ldots, k-1\}$, take

$$Q_j(u'_j, u'_{j+1}) = F'_j(u'_j, u_{j+1}) \cup \bar{F}'_{j+1}(u_{j+1}, u'_{j+1}) = v_0^j v_1^j v_2^j \ldots v_{jp}^j.$$

For $j = 0$, let u'_0 be the predecessor of x_2 on the path $P_2(u_0, x_2)$. Set

$$Q_0(u'_0, u'_1) = \bar{P}_2(u'_0, u_0) \cup F_0(u_0, u_1) \cup \bar{F}'_1(u_1, u'_1) = v_0^0 v_1^0 v_2^0 \ldots v_{0_p}^0.$$

For $j = k$, let v'_0 be the predecessor of x_1 on the path $P_1(u_0, x_1)$. Set

$$Q_k(u'_k, v'_0) = F'_k(u'_k, u_0) \cup P_1(u_0, v'_0) = v_0^k v_1^k v_2^k \ldots v_{k_p}^k.$$

Hence for any j with $0 \le j \le k$, we have $|V(Q_j)| \ge |V(C)| + 1$.

Claim 2.2. *$\{u'_0, u'_1, \ldots, u'_k\}$ is an independent set.*

Proof. Suppose to the contrary that $u'_i u'_j \in E(G)$ for some i and j with $0 \le i < j \le k$.

If $j \ne i+1$, since $F_i(u_i, u_{i+1})$ is a subpath of $Q_i(u'_i, u'_{i+1})$ and $F_j(u_j, u_{j+1})$ is a subpath of $Q_j(u'_j, u'_{j+1})$, we have a path $\bar{Q}_i(u_i, u'_i) \cup Q_j(u'_j, u_j)$ in $G - (V(C) \setminus \{u_i, u_j\})$, contrary to Claim 2.1.

Suppose now $j = i+1$. Then $Q_i(u'_i, u'_{i+1}) \cup u'_i$ is a cycle longer than C, a contradiction. $\square$

By a similar argument as in Claim 2.2, we can get the following claim.

Claim 2.3. *$\{v'_0, u'_1, \ldots, u'_k\}$ is an independent set.* $\square$

For the path $Q_0(u'_0, u'_1)$, we have the following claim.

Claim 2.4. *For the path Q_0, let $R_0 = G - V(Q_0)$. If $d(u) < id(u'_0)$ for every $u \in N_{R_0}(u'_0) \cup \{u'_0\}$, then there exists some vertex $v_b^0 \in N^-_{Q_0}(u'_0) \setminus \{u'_0\}$ such that $d(v_b^0) \ge id(u'_0)$.*

Proof. Let $d(u'_0) = l+1$ and $d_1 \le d_2 \le \cdots \le d_l \le d_{l+1} \le \cdots$ be the degree sequence of the vertices of $N_1(u'_0) \cup N_2(u'_0)$. Denote $m_2 = \min\{d(x) : x \in N_2(u'_0)\}$, $M_2 = \max\{d(x) : x \in N_2(u'_0)\}$. Since $id(u'_0) > d(u'_0)$, $id(u'_0)$ is equal to m_2 or d_{l+1} or d_l by the definition of implicit degrees.

Suppose that $id(u'_0) = m_2$. We have $u'_0 x_2 \in E(G)$ and $x_2 u_2 \in E(G)$. Thus, by the choice of C, we have $u'_0 u_2 \notin E(G)$. Hence $u_2 \in N^-_{Q_0}(u'_0) \setminus \{u'_0\}$ and $u_2 \in N_2(u'_0)$. Therefore $d(u_2) \ge m_2 = id(u'_0)$. Take $v_b^0 = u_2$.

If $id(u'_0) = d_{l+1}$, then $d_{l+1} > M_2$. We also have the following two facts:

$$N^-_{Q_0}(u'_0) \cup N_{R_0}(u'_0) \subseteq N_1(u'_0) \cup N_2(u'_0) \cup \{u'_0\},$$

$$|N^-_{Q_0}(u'_0) \setminus \{u'_0\}| + |N_{R_0}(u'_0)| \ge d(u'_0) - 1 = l.$$

Since $u'_0 u'_1 \notin E(G)$, $v_{l_{Q_0}(u'_0)+1} \in N_2(u'_0)$. Now we consider the $l+1$ vertices in $(N^-_{Q_0}(u'_0) \setminus \{u'_0\}) \cup N_{R_0}(u'_0) \cup \{v_{l_{Q_0}(u'_0)+1}\}$. These $l+1$ vertices belong to $N_1(u'_0) \cup N_2(u'_0)$. Hence there exists among them a vertex v_b^0 such that $d(v_b^0) \ge d_{l+1} = id(u'_0)$. Since $d_{l+1} > M_2$, $v_b^0 \ne v_{l_{Q_0}(u'_0)+1}$. Noting that $v_b^0 \notin N_{R_0}(u'_0)$, so $v_b^0 \in N^-_{Q_0}(u'_0) \setminus \{u'_0\}$.

If $id(u'_0) = d_l$, we only need to consider the l vertices in $(N^-_{Q_0}(u'_0) \setminus \{u'_0\}) \cup N_{R_0}(u'_0)$. Since they belong to $N_1(u'_0) \cup N_2(u'_0)$, there exists a vertex v^0_b in these l vertices such that $d(v^0_b) \geq d_l = id(u'_0)$. By the assumption, $v^0_b \notin N_{R_0}(u'_0)$. Hence $v^0_b \in N^-_{Q_0}(u'_0) \setminus \{u'_0\}$. □

For the path Q_k, we can get a claim similar to Claim 2.4.

Claim 2.5. *For the path Q_k, let $R_k = G - V(Q_k)$. If $d(v) < id(v'_0)$ for any $v \in N_{R_k}(v'_0) \cup \{v'_0\}$, then there exists some vertex $v^k_a \in N^+_{Q_k}(v'_0) \setminus \{v'_0\}$ such that $d(v^k_a) \geq id(v'_0)$.* □

The following lemma is a generalization of a lemma obtained in [5].

Lemma 2.1. *Let G be a 2-connected graph and $P = v_0v_1v_2 \dots v_p$ be a path of G such that $N_{G-V(P)}(v_0) = \emptyset$. If $d(v_0) < id(v_0)$ and $v_0v_p \notin E(G)$, then either*

(1) *there is some $v_j \in N^-_P(v_0)$ such that $d(v_j) \geq id(v_0)$; or*
(2) *$N^-_P(v_0) = N_P(v_0) \cup \{v_0\} \setminus \{L_P(v_0)\}$, and for any $v_j \in N^-_P(v_0)$, $d(v_j) < id(v_0)$ and for any $v_i \in N_2(v_0)$, $d(v_i) \geq id(v_0)$.*

Proof. This proof is very similar to the proof of Claim 2.4. Let $d(v_0) = l + 1$ and $d_1 \leq d_2 \leq \cdots \leq d_l \leq d_{l+1} \leq \cdots$ be the degree sequence of the vertices of $N_1(v_0) \cup N_2(v_0)$. Let $m_2 = \min\{d(u) : u \in N_2(v_0)\}$, $M_2 = \max\{d(u) : u \in N_2(v_0)\}$. Since $id(v_0) > d(v_0)$, $id(v_0)$ is equal to m_2 or d_{l+1} or d_l by the definition of an implicit degree.

If $id(v_0) = d_{l+1}$, then $d_{l+1} > M_2$. We also have the following two facts:

$$N^-_P(v_0) \subseteq N_1(v_0) \cup N_2(v_0) \cup \{v_0\},$$
$$|N^-_P(v_0) \setminus \{v_0\}| \geq d(v_0) - 1 = l.$$

Since $v_0v_p \notin E(G)$, $v_{l_P(v_0)+1} \in N_2(v_0)$. Now we consider the $l + 1$ vertices in $(N^-_P(v_0) \setminus \{v_0\}) \cup \{v_{l_P(v_0)+1}\}$. We can verify that these $l + 1$ vertices belong to $N_1(v_0) \cup N_2(v_0)$. Hence there exists a vertex v_j such that $d(v_j) \geq d_{l+1} = id(v_0)$. Since $d_{l+1} > M_2$, $v_j \neq v_{l_P(v_0)+1}$. Hence $v_j \in N^-_P(v_0) \setminus \{v_0\}$.

If $id(v_0) = d_l$, we only need to consider the l vertices in $(N^-_P(v_0) \setminus \{v_0\})$. Since they belong to $N_1(v_0) \cup N_2(v_0)$, there exists a vertex v_j in these l vertices such that $d(v_j) \geq d_l = id(v_0)$.

If $id(v_0) = m_2$, suppose that (1) is false. If there is some vertex $v_i \in N^-_P(v_0) \setminus \{v_0\}$ which is not adjacent to v_0, then $d(v_i) \geq m_2 = id(v_0)$, a contradiction. Hence every vertex in $N^-_P(v_0) \setminus \{v_0\}$ is adjacent to v_0. We have also $d(x) < id(v_0)$ for every $x \in N^-_P(v_0)$. Obviously, $d(v_j) \geq id(v_0)$ for each $v_j \in N_2(v_0)$. Now (2) holds. □

We also have the following claim.

Claim 2.6. *For the path Q_j defined as before, we have*

(a) *$N^-_{Q_j}(u'_j) \neq N_{Q_j}(u'_j) \cup \{u'_j\} \setminus \{L_{Q_j}(u'_j)\}$ for each $j \in \{1, 2, \dots, k\}$,*
(b) *$N^+_{Q_j}(u'_{j+1}) \neq N_{Q_j}(u'_{j+1}) \cup \{u'_{j+1}\} \setminus \{L_{Q_j}(u'_{j+1})\}$ for each $j \in \{0, 1, \dots, k-1\}$.*

Proof. For each $j \in \{1, 2, \dots, k\}$, let u^+_j is the successor of u_j on C.

We first prove (a). Consider a case that $u'_j = u_j$. When $j \neq k$, since $u_ju_0 \notin E(G)$ and $u_jx_j \in E(G)$, and x_j lies after u_0 on the path Q_j, $N^-_{Q_j}(u_j) \neq N_{Q_j}(u_j) \cup \{u_j\} \setminus \{L_{Q_j}(u_j)\}$. When $j = k$, since $u_ku_1 \notin E(G)$ and $u_kx_k \in E(G)$, and x_k lies after u_1 on the path Q_k, $N^-_{Q_k}(u_k) \neq N_{Q_k}(u_k) \cup \{u_k\} \setminus \{L_{Q_k}(u_k)\}$.

Now consider the case that $u'_j \neq u_j$. Suppose to the contrary that there exists some $j \in \{1, 2, \dots, k\}$ such that $N^-_{Q_j}(u'_j) = N_{Q_j}(u'_j) \cup \{u'_j\} \setminus \{L_{Q_j}(u'_j)\}$. Since C is a longest cycle of G, $u'_ju^+_j \notin E(G)$. Then the only possibility of the situation $N^-_{Q_j}(u'_j) = N_{Q_j}(u'_j) \cup \{u'_j\} \setminus \{L_{Q_j}(u'_j)\}$ is that $L_{Q_j}(u'_j)$ lies before u^+_j on the path Q_j. By the 2-connectivity of G and by the choice of Q_j, we know that there exist some $v^j_s \in N^-_{Q_j}(u'_j) \setminus \{u'_j\}$ and some $v^j_t \in Q_j(v_{l_{Q_j}(u'_j)+1}, u_{j+1})$ such that $v^j_sv^j_t \in E(G)$. Now we consider the path

$$\tilde{F}_j(v^j_s, u_{j+1}) = v^j_sv^j_{s-1} \dots v^j_0v^j_{s+1}v^j_{s+2} \dots u_{j+1}.$$

Since $l_{\tilde{F}_j}(v^j_s) \geq t > l_{F'_j}(u'_j)$ and $\tilde{F}_j$ contains F_j as a subpath, it is contrary to the choice of F'_j. Hence $N^-_{Q_j}(u'_j) \neq N_{Q_j}(u'_j) \cup \{u'_j\} \setminus \{L_{Q_j}(u'_j)\}$.

Now we prove (b). Consider a case that $u'_{j+1} = u_{j+1}$. When $j \neq 0$, then $u_{j+1}u_0 \notin E(G)$ and $u_{j+1}x_{j+1} \in E(G)$ and x_{j+1} lies before u_0 on the path Q_j. Therefore $N^+_{Q_j}(u_{j+1}) \neq N_{Q_j}(u_{j+1}) \cup \{u_{j+1}\} \setminus \{L_{Q_j}(u_{j+1})\}$. If $j = 0$, then $u_1u_2 \notin E(G)$ and $u_1x_1 \in E(G)$ and x_1 lies before u_2 on the path Q_0, which implies that $N^+_{Q_0}(u_1) \neq N_{Q_0}(u_1) \cup \{u_1\} \setminus \{L_{Q_0}(u_1)\}$.

Now consider the case that $u'_{j+1} \neq u_{j+1}$. Suppose to the contrary that there exists some $j \in \{0, 1, \dots, k-1\}$ such that $N^+_{Q_j}(u'_{j+1}) = N_{Q_j}(u'_{j+1}) \cup \{u'_{j+1}\} \setminus \{L_{Q_j}(u'_{j+1})\}$. Since C is a longest cycle of G, $u'_{j+1}u^+_{j+1} \notin E(G)$. Hence the only possibility of the situation $N^+_{Q_j}(u'_{j+1}) = N_{Q_j}(u'_{j+1}) \cup \{u'_{j+1}\} \setminus \{L_{Q_j}(u'_{j+1})\}$ is that $L_{Q_j}(u'_{j+1})$ lies after u^+_{j+1} on the path Q_j. By the 2-connectivity of G and by the choice of Q_j, we know that there exist some $v^j_c \in N^+_{Q_j}(u'_{j+1}) \setminus \{u'_{j+1}\}$ and some $v^j_d \in Q_j(u'_j, v_{l_{Q_j}(u'_{j+1})-1})$ such that $v^j_c v^j_d \in E(G)$.

Since C is a longest cycle, if $j \neq 0$, then $v^j_d \notin V(P_{j+1}) \cup (V(P_j) \setminus \{x_j\}) \cup V(Q_j(u'_j, u_j))$. If $j = 0$, then $v^0_d \notin V(P_2(u_0, u'_0)) \cup V(P_1)$. By the definition of Q_j, we know that $v^j_d \in V(C) \cup V(Q_{j+1}(u'_{j+1}, u_{j+1}))$. Hence v^j_d lies on Q_{j+1}. We also know that $v^j_c \in V(Q_{j+1}(u'_{j+1}, u_{j+1}))$. Now we consider the path Q_{j+1}. Without loss of generality, we let the corresponding vertices of v^j_c and v^j_d on the path Q_{j+1} are $v^{j+1}_{c'}$ and $v^{j+1}_{d'}$. Here we can get a new path

$$\hat{F}_{j+1}(v^{j+1}_{c'}, u_{j+2}) = v^{j+1}_{c'} v^{j+1}_{c'-1} \dots v^{j+1}_0 v^{j+1}_{c'+1} v^{j+1}_{c'+2} \dots u_{j+2}.$$

Since $l_{\hat{F}_{j+1}}(v^{j+1}_{c'}) \geq d' > l_{F'_{j+1}}(u'_{j+1})$ and $\hat{F}_{j+1}$ contains F_{j+1} as a subpath, it is contrary to the choice of F'_{j+1}. $\square$

In the next part of the proof we will look for a new path $W_j(w^j_1, w^j_2)$ corresponding to $Q_j(u'_j, u'_{j+1})$ for each $0 \leq j \leq k-1$ such that $V(Q_j) \subseteq V(W_j)$ and $d(w^j_1) \geq id(u'_j)$ and $d(w^j_2) \geq id(u'_{j+1})$. For $j = k$, we will look for a new path $W_k(w^k_1, w^k_2)$ corresponding to $Q_k(u'_k, v'_0)$ such that $V(Q_k) \subseteq V(W_k)$ and $d(w^k_1) \geq id(u'_k)$ and $d(w^k_2) \geq id(v'_0)$.

Fix $j \in \{0, \dots, k\}$. Consider the path $Q_j(u'_j, u'_{j+1})$.

Case 1. $d(u'_j) = id(u'_j)$.

Subcase 1.1. $j \neq k$.

If $d(u'_{j+1}) = id(u'_{j+1})$, take

$$W_j(w^j_1, w^j_2) = Q_j(u'_j, u'_{j+1}), \quad (w^j_1 = u'_j,\ w^j_2 = u'_{j+1}).$$

Suppose $d(u'_{j+1}) < id(u'_{j+1})$. By Claim 2.6, we can know $N^+_{Q_j}(u'_{j+1}) \neq N_{Q_j}(u'_{j+1}) \cup \{u'_{j+1}\} \setminus \{L_{Q_j}(u'_{j+1})\}$. Then there exists some vertex $v^j_m \in N^+_{Q_j}(u'_{j+1}) \setminus \{u'_{j+1}\}$ such that $d(v^j_m) \geq id(u'_{j+1})$ by Lemma 2.1 applied to the path $\bar{Q}_j(u'_{j+1}, u'_j)$. Now take

$$W_j(w^j_1, w^j_2) = v^j_0 v^j_1 \dots v^j_{m-1} v^j_{j_p} v^j_{j_p-1} \dots v^j_m, \quad (w^j_1 = u'_j,\ w^j_2 = v^j_m).$$

Subcase 1.2. $j = k$.

If there exists some vertex v in $N_{G-V(Q_k)}(v'_0) \cup \{v'_0\}$ such that $d(v) \geq id(v'_0)$, take

$$W_k(w^k_1, w^k_2) = Q_k(u'_k, v'_0) \cup v, \quad (w^k_1 = u'_k,\ w^k_2 = v).$$

Suppose $d(v) < id(v'_0)$ for every vertex v in $N_{G-V(Q_k)}(v'_0) \cup \{v'_0\}$. By Claim 2.5, there exists some v^k_a in $N^+_{Q_k}(v'_0) \setminus \{v'_0\}$ such that $d(v^k_a) \geq id(v'_0)$. Take

$$W_k(w^k_1, w^k_2) = v^k_0 v^k_1 \dots v^k_{a-1} v^k_{k_p} v^k_{k_p-1} \dots v^k_a, \quad (w^k_1 = u'_k,\ w^k_2 = v^k_a).$$

Case 2. $d(u'_j) < id(u'_j)$.

Subcase 2.1. $j = 0$ and there is some vertex u in $N_{G-V(Q_0)}(u'_0)$ such that $d(u) \geq id(u'_0)$.

If $d(u'_1) = id(u'_1)$, take

$$W_0(w^0_1, w^0_2) = u \cup Q_0(u'_0, u'_1), \quad (w^0_1 = u,\ w^0_2 = u'_1).$$

Suppose $d(u'_1) < id(u'_1)$. By Claim 2.6, we know that $N^+_{Q_0}(u'_1) \neq N_{Q_0}(u'_1) \cup \{u'_1\} \setminus \{L_{Q_0}(u'_1)\}$. Using Lemma 2.1, there is some vertex v^0_m in $N^+_{Q_0}(u'_1) \setminus \{u'_1\}$ such that $d(v^0_m) \geq id(u'_1)$. Take

$$W_0(w^0_1, w^0_2) = u v^0_0 v^0_1 \dots v^0_{m-1} v^0_{0_p} v^0_{0_p-1} \dots v^0_m, \quad (w^0_1 = u,\ w^0_2 = v^0_m).$$

Subcase 2.2. $j = 0$ and $d(u) < id(u'_0)$ for each u in $N_{G-V(Q_0)}(u'_0)$.

By Claim 2.4, there exists some vertex $v^0_b \in N^-_{Q_0}(u'_0) \setminus \{u'_0\}$ such that $d(v^0_b) \geq id(u'_0)$.

If $d(u'_1) = id(u'_1)$, take

$$W_0(w^0_1, w^0_2) = v^0_b v^0_{b-1} \dots v^0_0 v^0_{b+1} v^0_{b+2} \dots v^0_{0_p}, \quad (w^0_1 = v^0_b,\ w^0_2 = u'_1).$$

Suppose $d(u'_1) < id(u'_1)$. If $b + 1 \leq l_{Q_0}(u'_1)$ (where b is the index of the vertex v^0_b on the path Q_0), by Lemma 2.1 and Claim 2.6, there exists some vertex v^0_m in $N^+_{Q_0}(u'_1) \setminus \{u'_1\}$ such that $d(v^0_m) \geq id(u'_1)$. Take

$$W_0(w^0_1, w^0_2) = v^0_b v^0_{b-1} \dots v^0_0 v^0_{b+1} v^0_{b+2} \dots v^0_{m-1} v^0_{0_p} v^0_{0_p-1} \dots v^0_m, \quad (w^0_1 = v^0_b,\ w^0_2 = v^0_m).$$

If $b+1 > l_{Q_0}(u'_1)$, let

$$A_0 = \{v^0_x : v^0_x \in N^-_{Q_0}(u'_1) \text{ and } x < b\}, \qquad B_0 = \{v^0_y : v^0_y \in N^+_{Q_0}(u'_1) \text{ and } y > b+1\},$$

and

$$C_0 = \{v^0_z : v^0_z \in N^+_{Q_0}(u'_1),\ z < b+1 \text{ and } z \text{ is as large as possible}\}.$$

Then $|A_0| + |B_0| + |C_0| - |\{u'_1\}| \geq d(u'_1)$ and $v_{l_{Q_0}(u'_1)-1} \in A_0 \cap N_2(u'_1)$, and $v^0_z \in N_2(u'_1)$. By the definition of implicit degrees and a similar proof of Claim 2.4, we can get that there exists a vertex $v^0_h \in (A_0 \cup B_0) \setminus \{u'_1\}$ such that $d(v^0_h) \geq id(u'_1)$. When $v^0_h \in B_0 \setminus \{u'_1\}$, take

$$W_0(w^0_1, w^0_2) = v^0_b v^0_{b-1} \dots v^0_0 v^0_{b+1} v^0_{b+2} \dots v^0_{h-1} v^0_{0_p} v^0_{0_p-1} \dots v^0_h, \quad (w^0_1 = v^0_b,\ w^0_2 = v^0_h).$$

When $v^0_h \in A_0$, take

$$W_0(w^0_1, w^0_2) = v^0_b v^0_{b-1} \dots v^0_{h+1} v^0_{0_p} v^0_{0_p-1} \dots v^0_{b+1} v^0_0 v^0_1 \dots v^0_h, \quad (w^0_1 = v^0_b,\ w^0_2 = v^0_h).$$

Subcase 2.3. $j \neq 0$.

By Lemma 2.1 and Claim 2.6, we know that there exists a vertex $v^j_b \in N^-_{Q_j}(u'_j) \setminus \{u'_j\}$ such that $d(v^j_b) \geq id(u'_j)$. We also have the following two cases.

a. $j \neq k$.

If $d(u'_{j+1}) = id(u'_{j+1})$, take

$$W_j(w^j_1, w^j_2) = v^j_b v^j_{b-1} \dots v^j_0 v^j_{b+1} v^j_{b+2} \dots v^j_{j_p}, \quad (w^j_1 = v^j_b,\ w^j_2 = u'_{j+1}).$$

Suppose $d(u'_{j+1}) < id(u'_{j+1})$. If $b+1 \leq l_{Q_j}(u'_{j+1})$ (where b is the index of vertex v^j_b on the path Q_j), by Lemma 2.1 and Claim 2.6, we can obtain that there has some vertex v^j_m in $N^+_{Q_j}(u'_{j+1}) \setminus \{u'_{j+1}\}$ such that $d(v^j_m) \geq id(u'_{j+1})$. Take

$$W_j(w^j_1, w^j_2) = v^j_b v^j_{b-1} \dots v^j_0 v^j_{b+1} v^j_{b+2} \dots v^j_{m-1} v^j_{j_p} v^j_{j_p-1} \dots v^j_m, \quad (w^j_1 = v^j_b, w^j_2 = v^j_m).$$

If $b+1 > l_{Q_j}(u'_{j+1})$, similar to the proof of Subcase 2.2, we know that there exists some vertex $v^j_h \in (A_j \cup B_j) \setminus \{u'_{j+1}\}$ such that $d(v^j_h) \geq id(u'_{j+1})$. When $v^j_h \in B_j \setminus \{u'_{j+1}\}$, take

$$W_j(w^j_1, w^j_2) = v^j_b v^j_{b-1} \dots v^j_0 v^j_{b+1} v^j_{b+2} \dots v^j_{h-1} v^j_{j_p} v^j_{j_p-1} \dots v^j_h, \quad (w^j_1 = v^j_b, w^j_2 = v^j_h).$$

When $v^j_h \in A_j$, take

$$W_j(w^j_1, w^j_2) = v^j_b v^j_{b-1} \dots v^j_{h+1} v^j_{j_p} v^j_{j_p-1} \dots v^j_{b+1} v^j_0 v^j_1 \dots v^j_h, \quad (w^j_1 = v^j_b, w^j_2 = v^j_h).$$

b. $j = k$.

If there exists some vertex $v \in N_{G-V(Q_k)}(v'_0) \cup \{v'_0\}$ such that $d(v) \geq id(v'_0)$, take

$$W_k(w^k_1, w^k_2) = v^k_b v^k_{b-1} \dots v^k_0 v^k_{b+1} v^k_{b+2} \dots v^k_{k_p} v, \quad (w^k_1 = v^k_b, w^k_2 = v).$$

Suppose $d(v) < id(v'_0)$ for each v in $N_{G-V(Q_k)}(v'_0) \cup \{v'_0\}$. If $b+1 \leq l_{Q_k}(v'_0)$ (where b is the index of vertex v^k_b on the path Q_k), by Claim 2.5, there has some vertex v^k_a in $N^+_{Q_k}(v'_0) \setminus \{v'_0\}$ such that $d(v^k_a) \geq id(v'_0)$. Take

$$W_k(w^k_1, w^k_2) = v^k_b v^k_{b-1} \dots v^k_0 v^k_{b+1} v^k_{b+2} \dots v^k_{a-1} v^k_{k_p} v^k_{k_p-1} \dots v^k_a, \quad (w^k_1 = v^k_b, w^k_2 = v^k_a).$$

If $b+1 > l_{Q_k}(v'_0)$, let

$$A_k = \{v^k_x : v^k_x \in N^-_{Q_k}(v'_0) \text{ and } x < b\}, \qquad B_k = \{v^k_y : v^k_y \in N^+_{Q_k}(v'_0) \text{ and } y > b+1\},$$

and

$$C_k = \{v^k_z : v^k_z \in N^+_{Q_k}(v'_0),\ z < b+1 \text{ and } z \text{ is as large as possible}\}.$$

Then $|A_k| + |B_k| + |C_k| + |N_{G-V(Q_k)}(v'_0)| - |\{v'_0\}| \geq d(v'_0)$ and $v_{l_{Q_k}(v'_0)-1} \in A_k \cap N_2(v'_0)$ and $v^k_z \in N_2(v'_0)$. By the definition of implicit degrees and a similar proof of Claim 2.4, we know that there is some vertex $v^k_h \in (A_k \cup B_k) \setminus \{v'_0\}$ such that $d(v^k_h) \geq id(v'_0)$. When $v^k_h \in B_k \setminus \{v'_0\}$, take

$$W_k(w^k_1, w^k_2) = v^k_b v^k_{b-1} \dots v^k_0 v^k_{b+1} v^k_{b+2} \dots v^k_{h-1} v^k_{k_p} v^k_{k_p-1} \dots v^k_h, \quad (w^k_1 = v^k_b, w^k_2 = v^k_h).$$

When $v_h^k \in A_k$, take

$$W_k(w_1^k, w_2^k) = v_b^k v_{b-1}^k \dots v_{h+1}^k v_{k_p}^k v_{k_p-1}^k \dots v_{b+1}^k v_0^k v_1^k \dots v_h^k, \quad (w_1^k = v_b^k, w_2^k = v_h^k).$$

By the above discussion, we know that $W_j(w_1^j, w_2^j)$ contains at least $|V(C)| + 1$ vertices for each j with $0 \le j \le k$. Since C is a longest cycle of G, $w_1^j w_2^j \notin E(G)$, and

$$N(w_1^j) \cap N(w_2^j) \cap (V(G) \setminus V(W_j)) = \emptyset \quad \text{and} \quad N_{W_j}^-(w_1^j) \cap N_{W_j}(w_2^j) = \emptyset.$$

Hence

$$d(w_1^j) + d(w_2^j) \le |V(G) \setminus V(W_j)| + (|V(W_j)| - 1) = n - 1.$$

Therefore

$$\sum_{j=0}^{k} (d(w_1^j) + d(w_2^j)) \le (k+1)(n-1).$$

On the other hand, by the above discussion, we have

$$\begin{aligned}
\sum_{j=0}^{k} (d(w_1^j) + d(w_2^j)) &\ge \sum_{j=0}^{k-1} (id(u_j') + id(u_{j+1}')) + id(u_k') + id(v_0') \\
&\ge 2\sum_{j=1}^{k} id(u_j') + 2\min\{id(u_0'), id(v_0')\} \\
&\ge 2\sigma_{k+1}^*(G) > (k+1)(n-1),
\end{aligned}$$

which is a contradiction.

The proof is completed. □

References

[1] J.A. Bondy, Longest paths and cycles in graphs with high degree, Research Report CORR 80–16, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada, 1980.
[2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Elsevier, New York, 1976.
[3] G.A. Dirac, Some theorems on abstract graphs, Proc. Lond. Math. Soc. 3 (1952) 69–81.
[4] O. Ore, Note on Hamilton circuits, Amer. Math. Monthly 67 (1960) 55.
[5] Y. Zhu, H. Li, X. Deng, Implicit-degrees and circumferences, Graphs Combin. 5 (1989) 283–290.